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Self-dressing and radiation reaction in classical electrodynamics

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Abstract
A canonical approach to self-dressing in classical electrodynamics is presented. A slowly moving, rigid charge distribution is assumed to be completely deprived of the transverse electric field $\boldsymbol{E}_\perp$ at an initial time t_1' and the development of this component of the field is studied for $t > t_1'$ by solving the coupled charge-field Hamilton equations of motion. The theory is specialized to charge distributions of spherical symmetry, and in particular the point-charge, the spherical shell of charge and the spherical volume of charge are considered. As for the dynamics of the charge, the radiation-reaction force during self-dressing is obtained and it is shown to be substantially different at short times from the familiar form obtained for fully dressed charges, although it reduces to the latter for times longer than the time taken by the light to traverse the charge. Finally the most prominent features of the solution of the charge equation of motion for short times are discussed. As for the field, an auxiliary field $\boldsymbol{E}_c$ is introduced which is related to $\boldsymbol{E}_\perp$ and which has the advantage of being easily calculable. It is shown that $\boldsymbol{E}_c$ propagates causally for all the charge distributions considered and the way in which $\boldsymbol{E}_\perp$ can be obtained from $\boldsymbol{E}_c$ is illustrated. In addition it is shown that the radiation-reaction force is very simply related to the force exerted on the charge by $\boldsymbol{E}_c$ alone. In this way the details of the time dependence of the radiation-reaction force can be understood in terms of the behaviour of the field during self-dressing. It is argued that the results obtained for the classical model are capable of shedding light on fundamental issues of quantum electrodynamics, such as the theory of measurement of the field amplitude and the onset of irreversible behaviour during self-dressing.

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1. Introduction

The conceptual difficulty of separating sources from interacting fields in quantum field theory has led in the past to the notion of dressing the bare microscopic building blocks of matter (indicated as partons in an appropriate context [1]) by one or more zero-point radiation fields. Thus in many circumstances it is convenient to focus on composite quantum mechanical systems such as an atom or a molecule dressed by a cloud of virtual photons (in quantum optics [2]), an electron or an exciton dressed by a cloud of virtual phonons (in solid state physics [3]), an electron dressed by a cloud of virtual electron–positron pairs (in relativistic QED [4]), a nucleon dressed by a cloud of virtual π-mesons (in meson physics [5]) and quark–antiquark pairs dressed by gluons (in QCD [6]). The mathematical structure of the theory of dressed sources has been investigated in depth by Van Hove [7]. The dressed systems are usually taken in a steady-state regime and often coincide with the total ground state of the source-field Hamiltonian. In contrast, situations where the virtual cloud at some prescribed time is not in the steady-state have also been considered and preliminarily discussed in terms of half-dressed sources [8]. Assuming that appropriate dissipative processes exist [9, 10], the asymptotic fate of an half-dressed source is to attain the fully dressed configuration. The quantum dynamics of self-dressing has been described in a series of papers and summarized in a recent review [11]. In all cases considered, taken from QED, solid state physics and meson physics, the reconstruction of the virtual cloud is causal and takes place at the speed of propagation of the field contributing to the cloud. Thus, although the experimental aspects of the subject remain largely unexplored, physical effects related to the virtual cloud, such as self-energy shifts, mass and charge renormalization and van der Waals forces [11], should be expected to display time dependence at least when probed at sufficiently short times. From this viewpoint it appears that reconstruction of the virtual cloud and self-dressing are related to the fundamental problems of the quantum theory such as causality in quantum field theory [12, 13], nonexponential decay of unstable particles [14] and measurement of the quantum field amplitude [15, 16].

The importance of these issues, as well as the fact that comparison of classical and quantum models often yields insight into physical phenomena [17], suggests considering a classical analogue for the reconstruction of the virtual cloud and for self-dressing. Several classical phenomena are reminiscent of self-dressing. An obvious candidate is transition radiation, which is emitted by a charge crossing the boundary between two media with uniform velocity [18]. In this case the radiation emitted may be thought to originate from the rearrangement of the steady-state field surrounding the charge, which takes place when the charge crosses the boundary. Perhaps less well known is that the mechanical counterpart of transition radiation exists [19] for sources travelling in inhomogeneous elastic systems and finds applications in railway engineering. Other examples can be found in the theory of earthquakes, according to which the strain energy released is originally concentrated in a rather localized region around a fault and, after rupturing, is in part radiated away in the form of seismic waves [20]. A further example, most relevant for the purposes of the present paper, is radiation reaction in classical electrodynamics [21] or in acoustics [22]. In particular the former can be interpreted as a dynamical manifestation of the electromagnetic field surrounding a charge [23]. In fact the radiation-reaction force stems from the rearrangement of this electromagnetic field simultaneous with changes in the velocity of the charge.

Most of the efforts dedicated for over a century to the phenomenon of radiation reaction in electrodynamics have focused on the form of the radiation-reaction force and on the dynamical effects on the motion of the charge [24]. In contrast, in this paper we concentrate on the dynamics of the field surrounding the charge. Naturally these two aspects are inseparable even in classical electrodynamics, because the coupling between the equations of motion for source and field is at the root of this phenomenon. Consequently we have to consider also the mechanical aspect to some extent. The link between the dynamics of the charge and that of the field was emphasized some time ago by Moniz and Sharp [25] who pointed out that, in obtaining as usual the expression of the radiation-reaction force in terms of the derivatives of the charge velocity, one has to eliminate the field degrees of freedom by specifying the radiation field at some definite time. The simplest state of the classical field is the completely unexcited configuration, in contrast with the quantum case where the zero-point field is always present. This suggests that one can investigate the classical self-dressing of a charge starting from the completely unexcited field configuration and follow the time development of both the field and radiation-reaction force. More precisely, in this paper we shall adopt a canonical approach for a rigid classical charge distribution coupled to the classical electromagnetic field in the radiation gauge. We shall discuss the motion of the charge in the nonrelativistic limit and take the amplitude of all the field normal modes as zero at a time which, for historical reasons connected with a previous investigation in the framework of the Bohr–Rosenfeld theory of the measurement of the field amplitude [16] we have chosen to indicate as t_1'. The Hamilton equations of motion for the charge, after elimination of the transverse field, yield the desired expression for the radiation-reaction force $F_{\mathrm{RR}}(t)$, whereas the Hamilton equations for the field amplitudes can be integrated in terms of the motion of the charge. The part of this procedure leading to $F_{\mathrm{RR}}(t)$ is not new [26], but here we shall use it to discuss short times after t_1'. In this way we shall be able to obtain the form of $F_{\mathrm{RR}}(t)$ and to relate it to the time development of the field, following in detail the various stages of the self-dressing process, starting from an initial bare configuration of the charge.

At this point the difference is evident between our approach and the usual treatment of radiation reaction, where the assumption of vanishing field mode amplitude at finite time t_1' is not made and the field at t_1' has the equilibrium value at each point in space appropriate to a particle moving with a given velocity. Consequently we cannot rely on the considerations developed by Rohrlich. In fact, as we shall see, the form of $F_{\mathrm{RR}}(t)$ during self-dressing is nontrivially different, for sufficiently short times after t_1', from the familiar expression valid for a fully dressed charge [28] and reduces to the latter only after the initial stage of self-dressing is completed. We shall briefly discuss the possibility that the effects of this difference can be detected experimentally. Finally we shall obtain explicit expressions for the time-dependent field dressing the charge and relate the form of this field to that of the radiation-reaction force during self-dressing.

2. Canonical approach to radiation reaction

In the Coulomb gauge the Hamiltonian for a set of identical point charges q_i of mass m_i located at $\boldsymbol{x}_i$ is, in Gauss units

$$H = \sum_i \frac{1}{2m_i} {p_i}^2 + \frac{1}{2} \sum_{ij} \frac{q_i q_j}{|\boldsymbol{x}_i - \boldsymbol{x}_j|} + \frac{1}{8\pi} \int \mathrm{d}^3\boldsymbol{x} \left[\frac{1}{c^2} \dot{\boldsymbol{A}}_\perp^2 + (\nabla \times \boldsymbol{A}_\perp)^2 \right] - \sum_i \left[\frac{1}{m_i c} q_i \boldsymbol{p}_i \cdot \boldsymbol{A}_\perp(\boldsymbol{x}_i) - \frac{1}{2m_i c^2} {q_i}^2 \boldsymbol{A}_\perp^2(\boldsymbol{x}_i) \right]. \tag{2.1}$$

Assuming that the charges are rigidly connected, $|\boldsymbol{x}_i - \boldsymbol{x}_j|$ is constant and the Coulomb term in (2.1) does not play any role in our model and can be discarded. The vector potential $\boldsymbol{A}_\perp$

can be developed in plane waves with periodic boundary conditions on the surface of a cube of side L. Thus

$$A_\perp(x) = \sum_{k_j} \sqrt{\frac{2\pi\hbar c^2}{L^3\omega_k}} \epsilon_{k_j} \left(a_{kj}\,\mathrm{e}^{\mathrm{i}k\cdot x} + a^*_{kj}\,\mathrm{e}^{-\mathrm{i}k\cdot x}\right) \tag{2.2}$$

where $\omega_k = ck$ and a_{kj} is the classical complex amplitude of the field mode of wavevector k and real polarization ϵ_{kj} $(j = 1, 2)$. The appearance of $\hbar$, introduced in (2.2) and in the rest of this paper only for convenience, should not obscure the fact that the treatment we present is purely classical throughout the paper. The canonical coordinates and their conjugate momenta are x_i, p_i and a_{kj}, $\mathrm{i}\hbar a^*_{kj}$ [29]. From the Hamilton equations

$$\dot{x}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial x_i} \tag{2.3}$$

we get

$$m_i\ddot{x}_i = \frac{1}{c}q_i\dot{x}_i \times H(x_i) - q_i \sum_{kj} \sqrt{\frac{2\pi\hbar}{L^3\omega_k}} \epsilon_{kj} \left(\dot{a}_{kj}\,\mathrm{e}^{\mathrm{i}k\cdot x_i} + \dot{a}^*_{kj}\,\mathrm{e}^{-\mathrm{i}k\cdot x_i}\right) \tag{2.4}$$

where $H = \nabla \times A_\perp$ is the magnetic field. In addition the Hamilton equations

$$\dot{a}_{kj} = \frac{1}{\mathrm{i}\hbar}\frac{\partial H}{\partial a^*_{kj}} \tag{2.5}$$

yield

$$\dot{a}_{kj} = -\mathrm{i}\omega_k a_{kj} + \frac{\mathrm{i}}{\hbar}\sum_i q_i \sqrt{\frac{2\pi\hbar}{L^3\omega_k}} \epsilon_{kj} \cdot \dot{x}_i\,\mathrm{e}^{-\mathrm{i}k\cdot x_i} \tag{2.6}$$

Assuming that the motion of the charge consists of a rigid one-dimensional translation Q (no rotation) along a given direction, such that $x_i = x_{i0} + Q(t)$, expression (2.6) is equivalent to

$$\dot{a}_{kj} = -\mathrm{i}\omega_k a_{kj} + \frac{\mathrm{i}}{\hbar}\sqrt{\frac{2\pi\hbar}{L^3\omega_k}} \epsilon_{kj} \cdot \dot{Q}_i \sum_i q_i\,\mathrm{e}^{-\mathrm{i}k\cdot x_i}. \tag{2.7}$$

En passant we note that the inclusion of charge rotation has recently been shown to lead to a mathematically consistent and physically viable classical Lorentz electrodynamics also in the limit of vanishing bare charge [30]. This however is out of the scope of the present study. Further, we spread the point charges into a continuum of charge density $\rho(x, t)$ such that

$$\sum_i q_i\,\mathrm{e}^{-\mathrm{i}k\cdot x_i} = \int_V \mathrm{d}^3x\,\rho(x, t)\mathrm{e}^{-\mathrm{i}k\cdot x} \tag{2.8}$$

where V is the volume occupied by the charges at time t. We assume that V is much smaller than L^3 and entirely contained inside the quantization volume throghout the motion of the system. Thus defining $F(x, t) = \rho(x, t)/q$, where $q = \sum_i q_i$, we have

$$\sum_i q_i\,\mathrm{e}^{-\mathrm{i}k\cdot x_i} = q\int_{L^3} \mathrm{d}^3x\,F(x, t)\mathrm{e}^{-\mathrm{i}k\cdot x} = q\int_{L^3} \mathrm{d}^3x\,F(x - Q(t))\mathrm{e}^{-\mathrm{i}k\cdot x}. \tag{2.9}$$

In addition we expand $F(x)$ as

$$F(x) = \frac{1}{L^3}\sum_k f(k)\mathrm{e}^{\mathrm{i}k\cdot x} \qquad f(k) = \int_{L^3} \mathrm{d}^3x\,F(x)\mathrm{e}^{-\mathrm{i}k\cdot x}. \tag{2.10}$$

Hence

$$\sum_i q_i\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}_i} = q\frac{1}{L^3}\int_{L^3}\mathrm{d}^3\boldsymbol{x}\,\sum_{\boldsymbol{k}'} f(\boldsymbol{k}')\mathrm{e}^{\mathrm{i}(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{x}}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{Q}} = qf(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{Q}} \tag{2.11}$$

and from (2.7) we obtain

$$\dot a_{\boldsymbol{k}j} = -\mathrm{i}\omega_k a_{\boldsymbol{k}j} + \frac{\mathrm{i}}{\hbar}\sqrt{\frac{2\pi\hbar}{L^3\omega_k}}q\boldsymbol{\epsilon}_{\boldsymbol{k}j}\cdot\dot{\boldsymbol{Q}}f(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{Q}}. \tag{2.12}$$

The general solution of (2.12) is

$$a_{\boldsymbol{k}j}(t) = a_{\boldsymbol{k}j}(t_1')\mathrm{e}^{-\mathrm{i}\omega_k(t-t_1')} + \frac{\mathrm{i}}{\hbar}\sqrt{\frac{2\pi\hbar}{L^3\omega_k}}qf(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}\omega_k(t-t_1')}\boldsymbol{\epsilon}_{\boldsymbol{k}j}\cdot\int_{t_1'}^{t}\mathrm{d}t'\,\dot{\boldsymbol{Q}}(t')\mathrm{e}^{\mathrm{i}\omega_k(t'-t_1')}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{Q}(t')} \tag{2.13}$$

where t_1' is the time at which we take the transverse field to vanish. Thus $a_{\boldsymbol{k}j}(t_1') = 0$ and differentiation of (2.13) yields

$$\dot a_{\boldsymbol{k}j}(t)\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}_i} = \frac{\omega_k}{\hbar}\sqrt{\frac{2\pi\hbar}{L^3\omega_k}}qf(\boldsymbol{k})\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}_{i0}}\boldsymbol{\epsilon}_{\boldsymbol{k}j}\cdot\int_{t_1'}^{t}\mathrm{d}t'\,\dot{\boldsymbol{Q}}(t')\mathrm{e}^{\mathrm{i}\omega_k(t'-t)}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot[\boldsymbol{Q}(t')-\boldsymbol{Q}(t)]} + \frac{\mathrm{i}}{\hbar}\sqrt{\frac{2\pi\hbar}{L^3\omega_k}}qf(\boldsymbol{k})\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}_{i0}}\boldsymbol{\epsilon}_{\boldsymbol{k}j}\cdot\dot{\boldsymbol{Q}}(t). \tag{2.14}$$

Before substituting (2.14) in (2.4) we make the following approximations. First, we take $\dot Q \ll c$ and consistently we neglect the Lorentz term in (2.4). Second, we assume small displacements such that $\boldsymbol{k}\cdot[\boldsymbol{Q}(t')-\boldsymbol{Q}(t)] \sim 0$. Then (2.4) takes the form

$$m_i\ddot{\boldsymbol{x}}_i = -\frac{2\pi}{L^3}q\sum_{\boldsymbol{k}j}\frac{1}{\omega_k}\boldsymbol{\epsilon}_{\boldsymbol{k}j}f(\boldsymbol{k})q_i\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}_{i0}}\boldsymbol{\epsilon}_{\boldsymbol{k}j}\cdot\left[\omega_k\int_{t_1'}^{t}\mathrm{d}t'\,\dot{\boldsymbol{Q}}(t')\mathrm{e}^{\mathrm{i}\omega_k(t'-t)} + \mathrm{i}\dot{\boldsymbol{Q}}(t)\right] + \mathrm{cc} \tag{2.15}$$

where the small displacement assumption involves some limitation of the sum over $\boldsymbol{k}$ which we shall discuss later. Summation over i gives

$$M\ddot{\boldsymbol{Q}} = -\frac{4\pi}{L^3}q^2\sum_{\boldsymbol{k}j}f^2(\boldsymbol{k})\boldsymbol{\epsilon}_{\boldsymbol{k}j}\int_{t_1'}^{t}\mathrm{d}t'\,\boldsymbol{\epsilon}_{\boldsymbol{k}j}\cdot\dot{\boldsymbol{Q}}(t')\cos\omega_k(t'-t) \tag{2.16}$$

where $M = \sum_i m_i$ and where we have used (2.11) with $\boldsymbol{Q} = 0$. We note that, in the absence of external forces, the RHS of (2.16) is the radiation-reaction force $\boldsymbol{F}_{\mathrm{RR}}(t)$ acting on the charge. The well-known rules for polarization sums [2] lead to

$$\frac{1}{L^3}\sum_{\boldsymbol{k}j}f^2(\boldsymbol{k})\boldsymbol{\epsilon}_{\boldsymbol{k}j}\int_{t_1'}^{t}\mathrm{d}t'\,\boldsymbol{\epsilon}_{\boldsymbol{k}j}\cdot\dot{\boldsymbol{Q}}(t')\cos\omega_k(t'-t) = \frac{1}{L^3}\sum_{\boldsymbol{k}}f^2(\boldsymbol{k})\int_{t_1'}^{t}\mathrm{d}t'\,[\dot{\boldsymbol{Q}}(t') - \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}\cdot\dot{\boldsymbol{Q}}(t')]\cos\omega_k(t'-t) \tag{2.17}$$

where $\hat{\boldsymbol{k}}$ is the unit vector along $\boldsymbol{k}$. Specializing to the case of a spherically symmetric charge distribution, we find

$$f(\boldsymbol{k}) = \int_{L^3}\mathrm{d}^3\boldsymbol{x}\,F(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} = \frac{4\pi}{k}\int_0^\infty\mathrm{d}x\,xF(x)\sin kx \equiv f(k). \tag{2.18}$$

Consequently the angular parts of the sums in (2.17) are easily worked out as

$$\begin{aligned}\frac{1}{L^3}\sum_{\boldsymbol{k}} f^2(\boldsymbol{k})\cos\omega_k(t'-t) &= \frac{1}{2\pi^2}\int_0^\infty \mathrm{d}k\, k^2 f^2(k)\cos\omega_k(t'-t)\\ \frac{1}{L^3}\sum_{\boldsymbol{k}} f^2(\boldsymbol{k})\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}\cdot\dot{\boldsymbol{Q}}(t')\cos\omega_k(t'-t) &= \frac{1}{6\pi^2}\dot{\boldsymbol{Q}}(t')\int_0^\infty \mathrm{d}k\, k^2 f^2(k)\cos\omega_k(t'-t).\end{aligned} \tag{2.19}$$

Thus for the RHS of (2.16) we have

$$\boldsymbol{F}_{\mathrm{RR}}(t) = -\frac{4}{3\pi}q^2\int_{t_1'}^{t}\mathrm{d}t'\,\dot{\boldsymbol{Q}}(t')\int_0^\infty \mathrm{d}k\, k^2 f^2(k)\cos\omega_k(t'-t) \tag{2.20}$$

which is a relatively compact expression for the classical radiation-reaction force on a slow spherically symmetric charge which is bare (i.e. deprived of the transverse part of its field) at $t = t_1'$.

The following points should be noted.

(i) Expression (2.20) is unambiguously the radiation-reaction force only in the absence of other forces acting on the charge. In the presence of another force $\boldsymbol{F}_{\mathrm{ext}}$ the partition of the total force into $\boldsymbol{F}_{\mathrm{ext}}$ and $\boldsymbol{F}_{\mathrm{RR}}$ is not experimentally verifiable and consequently questionable [31]. One has to redefine $\boldsymbol{F}_{\mathrm{RR}}$ when dealing with cases where $\boldsymbol{F}_{\mathrm{ext}} \neq 0$. We shall disregard here the question whether the forces which bind the rigid charge distribution should be included in $\boldsymbol{F}_{\mathrm{ext}}$ or not. (ii) From expression (2.20) it is manifest that uniform motion is not in general a solution of the dynamical equations during self-dressing. We shall consider this aspect more closely in the next section. (iii) The small-displacement approximation made in (2.14) effectively limits the range of k in the integration appearing in (2.20). Consequently an upper cut-off k_c should be introduced, which is actually a function of $t - t_1'$. Thus in general the limit of $\boldsymbol{F}_{\mathrm{RR}}$ for $t = t_1'$, as evaluated from (2.20), should not be expected to yield the true radiation-reaction force at $t = t_1'$ (which must always vanish) except in cases where

$$\int_0^{k_c} \mathrm{d}k\, k^2 f^2(k)\cos\omega_k(t'-t)$$

is a well-behaved function of $t' - t$ which is not influenced much by the value of k_c. Note, however, that the small-displacement approximation is less restrictive than the electric dipole approximation $ka \sim 0$, which we have not used in this paper. Consequently we are entitled to consider times such that $t - t_1' < 2a/c$. We shall discuss this point in the course of the next section. In addition it should be noted that Galgani *et al* [32] did not make the small-displacement approximation in a similar model and solved numerically the equations for the mode amplitudes in order to obtain the time dependence of the radiation energy spectrum. Their study, however, was in connection with the Fermi–Ulam–Pasta problem and they did not obtain the time development of the electric field as we shall do in section 4.

3. Radiation reaction on a spherically symmetric charge

In this section we shall discuss the form taken by $\boldsymbol{F}_{\mathrm{RR}}$ for a number of spherically symmetric charge distributions.

3.1. Point charge

Including the treatment of a point charge in the context of the present paper can be considered questionable. In fact it is not clear that a point charge should be considered unambiguously the limit of a spherically symmetric charge distribution, as also nonspherically symmetric charges can be reduced to the point limit by appropriate procedures. In addition the classical theory of the electron suffers from dynamical pathologies, such as runaway solutions and noncausal behaviour, when its radius shrinks below the so-called classical electron radius [21]. This has been taken to indicate the existence of a length scale below which the validity of classical electrodynamics is questionable [25, 28, 33, 34]. In addition, several attempts at shedding light on the dynamics of a classical point charge in the presence of radiation reaction are still under way [30, 35–38] with the aim of providing a classical or semiclassical version of the electron, and this problem does not seem completely settled yet. In this paper we do not intend to model an electron. Nevertheless we shall find it useful to compare the results of an extended charge with those of a point charge; so we disregard the subtleties discussed in these studies and proceed naively from our results of the previous sections.

For a point charge we use $F(x) = \delta(x)/4\pi x^2$ [39] and consequently $f(k) = 1$ from (2.18). Substituting in (2.20) yields, for $t > t_1'$,

$$\begin{aligned}\boldsymbol{F}_{\mathrm{RR}}(t) &= -\frac{4}{3\pi}q^2\int_{t_1'}^{t}\mathrm{d}t'\,\dot{\boldsymbol{Q}}(t')\int_0^\infty \mathrm{d}k\,k^2\cos ck(t'-t)\\ &= \frac{4}{3\pi}\frac{1}{c^2}q^2\int_{t_1'}^{t}\mathrm{d}t'\dot{\boldsymbol{Q}}(t')\frac{\mathrm{d}^2}{\mathrm{d}t'^2}\int_0^\infty \mathrm{d}k\cos ck(t'-t)\\ &= \frac{4}{3c^3}q^2\int_{t_1'}^{t}\mathrm{d}t'\,\dot{\boldsymbol{Q}}(t')\delta''(t'-t) = \frac{4}{3c^3}q^2\left[-\ddot{\boldsymbol{Q}}(t)\delta(0)+\frac{1}{2}\dddot{\boldsymbol{Q}}(t)\right]\end{aligned} \tag{3.1}$$

which leads to the Abraham–Lorentz equation for the point charge, including the divergent mass correction [40] as well as the 4/3 factor, on which however we shall not dwell in this paper [41].

3.2. Spherical shell

Most calculations of $\boldsymbol{F}_{\mathrm{RR}}$ have been performed on this model [24], which has the advantage of being mathematically manageable. In this model the charge is uniformly distributed over the surface of a sphere of radius a. Thus from (2.18) we find

$$F(x) = \frac{1}{4\pi a^2}\delta(x-a) \qquad f(k) = \frac{4\pi}{k}\int_0^\infty \mathrm{d}x\,\frac{x}{4\pi a^2}\delta(x-a)\sin kx = \frac{\sin ka}{ka} \tag{3.2}$$

and, after some algebra,

$$\int_0^\infty \mathrm{d}k\,k^2 f^2(k)\cos\omega_k(t'-t) = \frac{\pi}{4a^2c}\left[2\delta(t'-t)-\delta\left(\frac{2a}{c}+t'-t\right)-\delta\left(\frac{2a}{c}-t'+t\right)\right]. \tag{3.3}$$

Substitution of (3.3) into (2.20) leads, for $t > t_1'$, to

$$\boldsymbol{F}_{\mathrm{RR}}(t) = -\frac{1}{3a^2c}q^2\left[\dot{\boldsymbol{Q}}(t)-\dot{\boldsymbol{Q}}\left(t-\frac{2a}{c}\right)\theta\left(t-t_1'-\frac{2a}{c}\right)\right]. \tag{3.4}$$

We note that (3.4) reduces to the usual expression [28]

$$\boldsymbol{F}_{\mathrm{RR}}(t) = -\frac{1}{3a^2c}q^2\left[\dot{\boldsymbol{Q}}(t)-\dot{\boldsymbol{Q}}\left(t-\frac{2a}{c}\right)\right] \tag{3.5}$$

only for $t - t_1' > \frac{2a}{c}$, which we consider as an indication of the influence of retardation of the self-dressing process for $t - t_1' < \frac{2a}{c}$. Actually the discrepancy between (3.4) and (3.5) can be ascribed to the different assumptions about the initial state of the field. In fact (3.5) is obtained starting from the proper frame in which the charge is instantaneously at rest. In this proper frame the initial transverse field vanishes, while in the laboratory frame it has the equilibrium value, at each point in space, appropriate to a charge moving with velocity $\dot{\boldsymbol{Q}}$. Consequently, in the absence of external forces, there is no radiation reaction on the charge at any time. In contrast, here we are concerned with an initially bare particle in the laboratory frame, which starts building up its own dressing transverse field immediately and consequently experiences the radiation-reaction force (3.4) at times $t < t_1' + \frac{2a}{c}$ even in the absence of external forces. After time $t_1' + \frac{2a}{c}$ the dressing field in the immediate vicinity of the charge is completely reconstructed and (3.4) becomes equivalent to (3.5).

3.3. Spherical volume

Consideration of this model seems appropriate in view of a recent debate in connection with the Bohr–Rosenfeld theory of measurement of the quantum electromagnetic field amplitude [15, 16, 42, 43]. The charge is uniformly distributed within a sphere of radius a. Thus we have

$$F(x) = \frac{3}{4\pi a^3}\theta(a - x) \qquad f(k) = \frac{3}{ka^3}\int_0^a \mathrm{d}x\, x \sin kx = \frac{3}{ka} j_1(ka). \tag{3.6}$$

After some algebra we obtain

$$\int_0^\infty \mathrm{d}k\, k^2 f^2(k) \cos\omega_k(t' - t) = \frac{3\pi}{2a^3}\left[1 - 3\frac{c(t - t')}{2a} + 2\frac{c^3(t - t')^3}{8a^3}\right]\theta\left[\frac{2a}{c} - (t - t')\right] \tag{3.7}$$

which leads to

$$\boldsymbol{F}_{\mathrm{RR}}(t) = -\frac{2}{a^3}q^2\int_{t_1'}^t \mathrm{d}t'\dot{\boldsymbol{Q}}(t')\left[1 - 3\frac{c(t - t')}{2a} + 2\frac{c^3(t - t')^3}{8a^3}\right]\theta\left[\frac{2a}{c} - (t - t')\right]. \tag{3.8}$$

Expression (3.8) shows that $\boldsymbol{F}_{\mathrm{RR}}$ is, as expected, invariant under replacement of $\boldsymbol{Q}(t)$ by $\boldsymbol{Q}(t) + \boldsymbol{Q}_0$, where $\boldsymbol{Q}_0$ is independent of t. This is a consequence of the translational invariance of the Lagrangian from which the Hamiltonian (2.1) can be derived.

As remarked at the end of section 2, the apparent discontinuity in (3.1) and (3.4) at $t = t_1'$ is a consequence of the small-displacement approximation, which we have performed in order to obtain (2.20). This discontinuity could be eliminated by introducing a cut-off k_c in the integrations over k. In this way the δ-functions appearing in (3.1) and (3.3) would smear into peaks of width k_c^{-1} and the discontinuity would disappear. This confirms that in reality $\boldsymbol{F}_{\mathrm{RR}}(t_1')$ vanishes and that the true expressions for $\boldsymbol{F}_{\mathrm{RR}}(t)$ are different from those evaluated in this section in a range of times such that $t - t_1' \leqslant (ck_c)^{-1}$. Since $k_c \gg \frac{1}{2a}$, however, failure of expressions (3.1), (3.4) and (3.8) is limited to times such that $t - t_1' \ll \frac{2a}{c}$ and it is negligible for our purposes. It should be mentioned that the above outlined procedure to eliminate the discontinuity of $\boldsymbol{F}_{\mathrm{RR}}$ at $t = t_1'$ is reminiscent of the trick used by Yaghjian, who introduced by hand an appropriate step function in order to eliminate preacceleration of a charge of finite extent [24], although the context seems rather different.

Although the dynamics of the charge is not central to the subject of this paper, it may be interesting to discuss briefly the consequences of the form of (3.5) and (3.8) on the motion of an extended charged body, disregarding the failure of these expressions for times such that $t - t_1' \ll \frac{2a}{c}$. In particular, we concentrate on the range of t such that $t - t_1' < \frac{2a}{c}$. For a spherical shell, expression (3.4) yields the following equation of motion

$$M\ddot{\boldsymbol{Q}}(t) = -\frac{1}{3a^2c}q^2\dot{\boldsymbol{Q}}(t) \tag{3.9}$$

where M is the (positive) bare mass of the shell. The solution of (3.9) is a damped motion with a damping time

$$\gamma^{-1} = \frac{3Ma^2c}{q^2} \tag{3.10}$$

which is capable of bringing the shell essentially to rest before time $2a/c$, provided $2q^2/3a \gg Mc^2$. On the other hand, for a spherical volume expression (3.8) yields

$$M\ddot{\boldsymbol{Q}}(t) = -\frac{2}{a^3}q^2 \int_{t_1'}^{t} \mathrm{d}t' \dot{\boldsymbol{Q}}(t') \left[1 - 3\frac{c(t-t')}{2a} + 2\frac{c^3(t-t')^3}{8a^3}\right] \tag{3.11}$$

which is also likely to describe a damping motion of nonlocal character, but which we have failed to solve exactly. However one can expand

$$\dot{\boldsymbol{Q}}(t') = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(t-t')^n \boldsymbol{Q}^{(n+1)}(t) \tag{3.12}$$

where $\boldsymbol{Q}^{(n)}(t)$ is the nth derivative of $\boldsymbol{Q}$ with respect to t. Substituting (3.12) into (3.11) and keeping only derivatives of $\boldsymbol{Q}(t)$ up to the second in the corresponding expansion of the integral leads to

$$M\ddot{\boldsymbol{Q}}(t) = -\kappa\dot{\boldsymbol{Q}}(t) - \delta M(t)\ddot{\boldsymbol{Q}}(t) \tag{3.13}$$

where

$$\begin{aligned} \kappa(t) &= \frac{4}{a^2c}q^2\frac{c(t-t_1')}{2a}\left[1 - \frac{3}{2}\frac{c(t-t_1')}{2a} + \frac{1}{2}\frac{c^3(t-t_1')^3}{8a^3}\right] \\ \delta M(t) &= -\frac{8}{ac^2}q^2\frac{c^2(t-t_1')^2}{4a^2}\left[\frac{1}{2} - \frac{c(t-t_1')}{2a} + \frac{2}{5}\frac{c^3(t-t_1')^3}{8a^3}\right]. \end{aligned} \tag{3.14}$$

Thus in (3.13) both the mass correction δM and the damping coefficient κ are time dependent, and its solution is

$$\dot{\boldsymbol{Q}}(t) = \dot{\boldsymbol{Q}}(t_1')\exp\left(-\int_{t_1'}^{t}\mathrm{d}t'\frac{\kappa(t')}{M+\delta M(t')}\right) \tag{3.15}$$

which describes acceleration or damping depending on the relative values of the physical parameters of the system, since the dressed mass can become negative for

$$4(\sqrt{3}-1)\frac{q^2}{a} > Mc^2. \tag{3.16}$$

It is not clear, however, under which conditions one can truncate the series in (3.12). Nevertheless, as mentioned earlier, a discussion of the dynamics of an extended charge is beyond the scope of the present paper, although it should be pointed out that such a discussion might lead to interesting results, particularly in the presence of a harmonic binding force [44].

4. Self-dressing of a spherically symmetric charge

In the Coulomb gauge the general expression for the transverse electric field is

$$\boldsymbol{E}_\perp(\boldsymbol{x},t) = -\frac{1}{c}\dot{\boldsymbol{A}}_\perp(\boldsymbol{x},t) = \mathrm{i}\sum_{\boldsymbol{k}j}\sqrt{\frac{2\pi\hbar\omega_k}{V}}\boldsymbol{\epsilon}_{\boldsymbol{k}j}\left(a_{\boldsymbol{k}j}(t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} - a^*_{\boldsymbol{k}j}(t)\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\right). \tag{4.1}$$

In the presence of a charge the $a_{\boldsymbol{k}j}$ amplitudes are determined by the charge and by the initial conditions. Assuming spherical charge symmetry and a bare charge at $t = t_1'$, from (2.13) we get, after summing over polarization,

$$\boldsymbol{E}_\perp(\boldsymbol{x},t) = -\frac{4\pi}{L^3}q\sum_{\boldsymbol{k}} f(k)\left\{\int_{t_1'}^{t}\mathrm{d}t'\,\dot{\boldsymbol{Q}}(t')\cos[\boldsymbol{k}\cdot\boldsymbol{x}+\omega_k(t'-t)]\right.$$
$$\left. -\hat{\boldsymbol{k}}\int_{t_1'}^{t}\mathrm{d}t'\,\hat{\boldsymbol{k}}\cdot\dot{\boldsymbol{Q}}(t')\cos[\boldsymbol{k}\cdot\boldsymbol{x}+\omega_k(t'-t)]\right\} \tag{4.2}$$

which can also be cast in the form

$$E_{\perp i}(\boldsymbol{x},t) = -\frac{4\pi}{L^3}q\sum_{\boldsymbol{k}} f(k)(\delta_{ij}-\hat{\boldsymbol{k}}_i\hat{\boldsymbol{k}}_j)\int_{t_1'}^{t}\mathrm{d}t'\,\dot{Q}_j(t')\cos[\boldsymbol{k}\cdot\boldsymbol{x}+\omega_k(t'-t)]. \tag{4.3}$$

The last expression prompts the introduction of an auxiliary field

$$\boldsymbol{E}_c(\boldsymbol{x},t) = -\frac{4\pi}{L^3}q\sum_{\boldsymbol{k}} f(k)\int_{t_1'}^{t}\mathrm{d}t'\,\dot{\boldsymbol{Q}}(t')\cos[\boldsymbol{k}\cdot\boldsymbol{x}+\omega_k(t'-t)] \tag{4.4}$$

whose transverse component is $\boldsymbol{E}_\perp(\boldsymbol{x},t)$ and whose longitudinal component is a vector field $\boldsymbol{E}_L(\boldsymbol{x},t)$ defined as

$$E_{Li}(\boldsymbol{x},t) = -\frac{4\pi}{L^3}q\sum_{\boldsymbol{k}} f(k)\hat{k}_i\hat{k}_j\int_{t_1'}^{t}\mathrm{d}t'\,\dot{Q}_j(t')\cos[\boldsymbol{k}\cdot\boldsymbol{x}+\omega_k(t'-t)]$$
$$= \frac{4\pi}{L^3}q\nabla_i\nabla_j\sum_{\boldsymbol{k}}\frac{1}{k^2}f(k)\int_{t_1'}^{t}\mathrm{d}t'\,\dot{Q}_j(t')\cos[\boldsymbol{k}\cdot\boldsymbol{x}+\omega_k(t'-t)] \tag{4.5}$$

where $\nabla = \partial/\partial\boldsymbol{x}$.

From (4.2) we have the relation

$$\boldsymbol{E}_c(\boldsymbol{x},t) = \boldsymbol{E}_\perp(\boldsymbol{x},t) + \boldsymbol{E}_L(\boldsymbol{x},t). \tag{4.6}$$

Naturally $\boldsymbol{E}_c(\boldsymbol{x},t)$ should not be confused with the total field $\boldsymbol{E}_{\mathrm{TOT}}(\boldsymbol{x},t)$ created by the charge. The latter is given by

$$\boldsymbol{E}_{\mathrm{TOT}}(\boldsymbol{x},t) = \boldsymbol{E}_\perp(\boldsymbol{x},t) + \boldsymbol{E}_\parallel(\boldsymbol{x},t). \tag{4.7}$$

Nevertheless, although $\boldsymbol{E}_{\mathrm{TOT}}(\boldsymbol{x},t) \neq \boldsymbol{E}_c(\boldsymbol{x},t)$, $\boldsymbol{E}_\perp(\boldsymbol{x},t)$ is the transverse part of $\boldsymbol{E}_{\mathrm{TOT}}(\boldsymbol{x},t)$ as well as of $\boldsymbol{E}_c(\boldsymbol{x},t)$. In fact, at least in principle, one can evaluate $\boldsymbol{E}_\perp(\boldsymbol{x},t)$ as

$$E_{\perp i}(\boldsymbol{x},t) = \int \mathrm{d}^3\boldsymbol{x}'\,E_{cj}(\boldsymbol{x}',t)\delta_{\perp ij}(\boldsymbol{x}-\boldsymbol{x}') \tag{4.8}$$

where $\delta_{\perp ij}$ is the transverse part of the δ-function [45], once $\boldsymbol{E}_c(\boldsymbol{x},t)$ is known. Consequently $\boldsymbol{E}_c(\boldsymbol{x},t)$ contains the same information as $\boldsymbol{E}_\perp(\boldsymbol{x},t)$, and its form (4.4) is easier to evaluate than (4.3). Thus in what follows we shall concentrate on the derivation of $\boldsymbol{E}_c(\boldsymbol{x},t)$ for each of the charge distributions considered in the previous section, as a convenient means of discussing the self-dressing process. Eventually we shall also show that it is possible to relate $\boldsymbol{E}_c(\boldsymbol{x},t)$ directly to $\boldsymbol{F}_{\mathrm{RR}}(t)$ during self-dressing.

4.1. Point charge

As already mentioned $f(k) = 1$ for a point charge. Thus, changing sums into integrals and performing standard angular integrations, we have

$$\frac{1}{L^3}\sum_{\boldsymbol{k}} f(k)\cos[\boldsymbol{k}\cdot\boldsymbol{x}+\omega_k(t'-t)] = \frac{1}{2\pi^2}\frac{1}{x}\int_0^\infty \mathrm{d}k\,k\sin kx\cos ck(t'-t)$$
$$= \frac{1}{4\pi c^2}\frac{1}{x}\left\{\delta'\left[t'-\left(t+\frac{x}{c}\right)\right]-\delta'\left[t'-\left(t-\frac{x}{c}\right)\right]\right\} \tag{4.9}$$

where δ' indicates differentiation of the δ-function with respect to t'. Substitution of (4.9) into (4.4) yields

$$\boldsymbol{E}_c(\boldsymbol{x}, t) = -\frac{1}{c^2}\frac{1}{x} q\ddot{\boldsymbol{Q}}\left(t - \frac{x}{c}\right)\theta\left(t - t_1' - \frac{x}{c}\right). \tag{4.10}$$

We see that self-dressing of a point charge, as described by the field $\boldsymbol{E}_c(\boldsymbol{x}, t)$, is a causal process since it takes place within a sphere of radius $c(t - t_1')$ and since $\boldsymbol{E}_c(\boldsymbol{x}, t)$ vanishes outside such a sphere. For this reason we call such a sphere the 'causality sphere'. For a point charge it is also relatively easy, as well as instructive, to derive an explicit expression for the longitudinal component $\boldsymbol{E}_L(\boldsymbol{x}, t)$ of $\boldsymbol{E}_c(\boldsymbol{x}, t)$. In fact from (4.5) we have, using partial integration

$$\begin{aligned} E_{Li}(\boldsymbol{x}, t) &= \frac{2}{\pi} q\nabla_i\nabla_j \int_{t_1'}^{t} \mathrm{d}t'\, \dot{Q}_j(t') \frac{1}{x} \int_0^\infty \mathrm{d}k\, \frac{1}{k} \sin kx \cos ck(t' - t) \\ &= \frac{2}{\pi} q\nabla_i\nabla_j \frac{1}{x} \left\{ Q_j(t') \int_0^\infty \mathrm{d}k \frac{1}{k} \sin kx \cos ck(t' - t) \Big|_{t_1'}^{t} \right. \\ &\quad \left. - \int_{t_1'}^{t} \mathrm{d}t' Q_j(t') \frac{\mathrm{d}}{\mathrm{d}t'} \int_0^\infty \mathrm{d}k \frac{1}{k} \sin kx \cos ck(t' - t) \right\}. \end{aligned} \tag{4.11}$$

In addition

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t'} \int_0^\infty \mathrm{d}k \frac{1}{k} \sin kx \cos ck(t' - t) &= -c \int_0^\infty \mathrm{d}k\, \sin kx \sin ck(t' - t) \\ &= -\frac{\pi}{2} \left\{ \delta\left[t' - \left(t + \frac{x}{c}\right)\right] - \delta\left[t' - \left(t - \frac{x}{c}\right)\right] \right\} \end{aligned} \tag{4.12}$$

which, after substitution in (4.11) and some algebra, leads to

$$E_{Li}(\boldsymbol{x}, t) = q\nabla_i\nabla_j \frac{1}{x} \left\{ [Q_j(t) - Q_j(t_1')] - \left[Q_j\left(t - \frac{x}{c}\right) - Q_j(t_1') \right] \theta\left(t - t_1' - \frac{x}{c}\right) \right\}. \tag{4.13}$$

Explicitly we have

$$\begin{aligned} E_{Li}(\boldsymbol{x}, t) &= -\frac{1}{x^3} q(\delta_{ij} - 3\hat{x}_i\hat{x}_j)[Q_j(t) - Q_j(t_1')] + q \left\{ \frac{1}{x^3}(\delta_{ij} - 3\hat{x}_i\hat{x}_j) \left[Q_j\left(t - \frac{x}{c}\right) \right.\right. \\ &\quad \left. - Q_j(t_1') \right] + \frac{1}{c}\frac{1}{x^2}(\delta_{ij} - 3\hat{x}_i\hat{x}_j)\dot{Q}_j\left(t - \frac{x}{c}\right) \\ &\quad \left. - \frac{1}{c^2}\frac{\hat{x}_i\hat{x}_j}{x}\ddot{Q}_j\left(t - \frac{x}{c}\right) \right\} \theta\left(t - t_1' - \frac{x}{c}\right) \\ &\quad + \text{terms in } \delta[x - c(t - t_1')] \text{ and in } \delta'[x - c(t - t_1')]. \end{aligned} \tag{4.14}$$

Expressions (4.10) and (4.14) can be substituted in (4.6) to yield an explicit expression for the transverse field in the form

$$\begin{aligned} E_{\perp i}(\boldsymbol{x}, t) &= E_{ci}(\boldsymbol{x}, t) - E_{Li}(\boldsymbol{x}, t) \\ &= \frac{1}{x^3} q(\delta_{ij} - 3\hat{x}_i\hat{x}_j)[Q_j(t) - Q_j(t_1')] - q \left\{ \frac{1}{x^3}(\delta_{ij} - 3\hat{x}_i\hat{x}_j) \left[Q_j\left(t - \frac{x}{c}\right) \right.\right. \\ &\quad \left. - Q_j(t_1') \right] + \frac{1}{c}\frac{1}{x^2}(\delta_{ij} - 3\hat{x}_i\hat{x}_j)\dot{Q}_j\left(t - \frac{x}{c}\right) \\ &\quad \left. + \frac{1}{c^2}\frac{1}{x}(\delta_{ij} - \hat{x}_i\hat{x}_j)\ddot{Q}_j\left(t - \frac{x}{c}\right) \right\} \theta\left(t - t_1' - \frac{x}{c}\right) \\ &\quad + \text{terms in } \delta[x - c(t - t_1')] \text{ and in } \delta'[x - c(t - t_1')]. \end{aligned} \tag{4.15}$$

We see that neither $\boldsymbol{E}_\perp(\boldsymbol{x}, t)$ nor $\boldsymbol{E}_L(\boldsymbol{x}, t)$ are causal, although they both vanish at $t = t_1'$ for any $x > 0$ as expected.

We complete our discussion of self-dressing of a point charge by evaluating the total electric field

$$\boldsymbol{E}_{\text{TOT}}(\boldsymbol{x}, t) = \boldsymbol{E}_\parallel(\boldsymbol{x}, t) + \boldsymbol{E}_\perp(\boldsymbol{x}, t). \tag{4.16}$$

The longitudinal field $\boldsymbol{E}_\parallel(\boldsymbol{x}, t)$ is simply the Coulomb field generated by the point charge and propagates instantaneously. It is given by

$$\boldsymbol{E}_\parallel(\boldsymbol{x}, t) = q\frac{\boldsymbol{R}(t)}{R^3(t)} \qquad \boldsymbol{R}(t) = \boldsymbol{x} - \boldsymbol{Q}(t). \tag{4.17}$$

In the linear approximation used throughout this paper

$$E_{\parallel i}(\boldsymbol{x}, t) = q\frac{\hat{x}_i}{x^2} - \frac{1}{x^3}q(\delta_{ij} - 3\hat{x}_i\hat{x}_j)Q_j(t). \tag{4.18}$$

Substitution of (4.15) and (4.18) in (4.16) yields

$$\begin{aligned} E_{\text{TOT}i}(\boldsymbol{x}, t) = {} & q\frac{\hat{x}_i}{x^2} - \frac{1}{x^3}q(\delta_{ij} - 3\hat{x}_i\hat{x}_j)Q_j(t_1')\left[1 - \theta\left(t - t_1' - \frac{x}{c}\right)\right] \\ & - q\left[\frac{1}{x^3}(\delta_{ij} - 3\hat{x}_i\hat{x}_j)Q_j\left(t - \frac{x}{c}\right) + \frac{1}{c}\frac{1}{x^2}(\delta_{ij} - 3\hat{x}_i\hat{x}_j)\dot{Q}_j\left(t - \frac{x}{c}\right)\right. \\ & \left. + \frac{1}{c^2}\frac{1}{x}(\delta_{ij} - \hat{x}_i\hat{x}_j)\ddot{Q}_j\left(t - \frac{x}{c}\right)\right]\theta\left(t - t_1' - \frac{x}{c}\right) \\ & + \text{terms in } \delta[x - c(t - t_1')] \text{ and in } \delta'[x - c(t - t_1')]. \end{aligned} \tag{4.19}$$

Thus at $t = t_1'$ only the Coulomb field $\boldsymbol{E}_\parallel(\boldsymbol{x}, t_1')$ of the charge is present, in accord with the initial conditions assumed. For $t > t_1'$, $\boldsymbol{E}_{\text{TOT}}(\boldsymbol{x}, t)$ consists of this original field as well as of other contributions which propagate causally. The latter causal contributions do not arise from $\boldsymbol{E}_\perp(\boldsymbol{x}, t)$ alone (which, as we have seen, does not propagate causally) but also, in part, from the longitudinal field at time t. Indeed if one cancels $\boldsymbol{E}_\parallel(\boldsymbol{x}, t_1')$ by introducing a fixed charge $-q$ at point $\boldsymbol{Q}(t_1')$, which compensates exactly the longitudinal field of the mobile charge q at time t_1', $\boldsymbol{E}_{\text{TOT}}(\boldsymbol{x}, t)$ is immediately seen from (4.19) to propagate causally. Furhermore, in the limit $t_1' \to -\infty$, $\boldsymbol{E}_{\text{TOT}}$ takes the form

$$\begin{aligned} E_{\text{TOT}i}(\boldsymbol{x}, -\infty) = {} & q\frac{\hat{x}_i}{x^2} - q\left[\frac{1}{x^3}(\delta_{ij} - 3\hat{x}_i\hat{x}_j)Q_j\left(t - \frac{x}{c}\right) + \frac{1}{c}\frac{1}{x^2}(\delta_{ij} - 3\hat{x}_i\hat{x}_j)\dot{Q}_j\left(t - \frac{x}{c}\right)\right. \\ & \left. + \frac{1}{c^2}\frac{1}{x}(\delta_{ij} - \hat{x}_i\hat{x}_j)\ddot{Q}_j\left(t - \frac{x}{c}\right)\right] \end{aligned} \tag{4.20}$$

which can be shown to coincide with the Lienard–Wiechert field in the linear approximation, as expected of a charge whose dressing process has been completed long ago. We conclude that the self-dressing of a point charge proceeds as expected, in spite of the anomalies that plague the dynamics of the charge.

4.2. Spherical shell

As we have argued, $\boldsymbol{E}_c(\boldsymbol{x}, t)$ provides a good representation of the self-dressing process, since the transverse field can be obtained from it using (4.8) and since it is also possible to get $\boldsymbol{E}_{\text{TOT}}(\boldsymbol{x}, t)$ from (4.7) once $\boldsymbol{E}_\parallel(\boldsymbol{x}, t)$ is evaluated from the Poisson equation. Thus here we shall evaluate only $\boldsymbol{E}_c(\boldsymbol{x}, t)$ for the spherical shell using (4.6). For this case we have

$f(k) = \sin ka/ka$ and consequently, after performing angular integrations, we get

$$\frac{1}{L^3}\sum_{\boldsymbol{k}} f(k)\cos[\boldsymbol{k}\cdot\boldsymbol{x}+\omega_k(t'-t)] = \frac{1}{2\pi^2 a}\frac{1}{x}\int_0^\infty \mathrm{d}k \sin ka \sin kx \cos ck(t'-t)$$
$$= -\frac{1}{8\pi a}\frac{1}{x}\{\delta[a+x+c(t'-t)] - \delta[a-x-c(t'-t)] + \delta[a+x-c(t'-t)] - \delta[a-x+c(t'-t)]\}. \quad (4.21)$$

Substituting in (4.4) and performing the time integration leads to

$$\boldsymbol{E}_c(\boldsymbol{x},t) = \frac{1}{2ac}q\frac{1}{x}\left\{\dot{\boldsymbol{Q}}\left(t-\frac{x+a}{c}\right)\theta[c(t-t_1')-(x+a)] - \dot{\boldsymbol{Q}}\left(t-\frac{x-a}{c}\right)\theta[c(t-t_1') - (x-a)]\theta(x-a) - \dot{\boldsymbol{Q}}\left(t+\frac{x-a}{c}\right)\theta[c(t-t_1')+(x-a)]\theta(a-x)\right\}. \quad (4.22)$$

Outside the shell $(x > a)$

$$\boldsymbol{E}_c(\boldsymbol{x},t) = \frac{1}{2ac}q\frac{1}{x}\left\{\dot{\boldsymbol{Q}}\left(t-\frac{x+a}{c}\right)\theta[c(t-t_1')-(x+a)] - \dot{\boldsymbol{Q}}\left(t-\frac{x-a}{c}\right)\theta[c(t-t_1')-(x-a)]\right\}. \quad (4.23)$$

Thus outside the shell $\boldsymbol{E}_c(\boldsymbol{x},t)$ behaves causally, in the sense that it vanishes outside an expanding sphere of radius $c(t-t_1')+a$. In addition it is structured, in the sense that in the region $c(t-t_1')-a < x < c(t-t_1')+a$ its form is different from that in the region $x < c(t-t_1')-a$. This effect is a result of the interference between the fields emitted by different parts of the shell and carries information on the structure of the shell at large distances from the source. Note that as $a \to 0$ expression (4.23) tends to

$$\boldsymbol{E}_c(\boldsymbol{x},t) = -\frac{1}{c^2}\frac{1}{x}q\left[\ddot{\boldsymbol{Q}}\left(t-\frac{x}{c}\right)\theta\left(t-t_1'-\frac{x}{c}\right)+\dot{\boldsymbol{Q}}\left(t-\frac{x}{c}\right)\delta\left(t-t_1'-\frac{x}{c}\right)\right] \quad (4.24)$$

which is identical to the corresponding field of a point charge (4.10), except on the surface of the causality sphere. Inside the shell $(x < a)$ expression (4.22) gives

$$\boldsymbol{E}_c(\boldsymbol{x},t) = \frac{1}{2ac}q\frac{1}{x}\left\{\dot{\boldsymbol{Q}}\left(t-\frac{a+x}{c}\right)\theta[c(t-t_1')-(a+x)] - \dot{\boldsymbol{Q}}\left(t-\frac{a-x}{c}\right)\theta[c(t-t_1')-(a-x)]\right\}. \quad (4.25)$$

Thus in the inner space $\boldsymbol{E}_c(\boldsymbol{x},t)$ takes the form

(a) $\boldsymbol{E}_c(\boldsymbol{x},t) = 0 \qquad \left(t-t_1' < \frac{a-x}{c}\right)$

(b) $\boldsymbol{E}_c(\boldsymbol{x},\mathrm{t}) = -\frac{1}{2ac}q\frac{1}{x}\dot{\boldsymbol{Q}}\left(t-\frac{a-x}{c}\right) \qquad \left(\frac{a-x}{\mathrm{c}} < t-t_1' < \frac{a+x}{c}\right)$

(c) $\boldsymbol{E}_c(\boldsymbol{x},\mathrm{t}) = \frac{1}{2ac}q\frac{1}{x}\left[\dot{\boldsymbol{Q}}\left(t-\frac{a+x}{c}\right)-\dot{\boldsymbol{Q}}\left(t-\frac{a-x}{c}\right)\right] \qquad \left(t-t_1' > \frac{a+x}{c}\right).$

(4.26)

Evidently $\boldsymbol{E}_c(\boldsymbol{x},t)$ also behaves causally inside the shell, in the sense that in a first phase it vanishes inside a contracting sphere of radius $a-c(t-t_1')$ and it is of the form (b) in (4.26) outside this causality sphere. This first phase ends at time $t = t_1' + \frac{a}{c}$, when the radius of the contracting causality sphere vanishes and another causality sphere of radius $c(t-t_1')$ arises. Inside this expanding causality sphere $\boldsymbol{E}_c(\boldsymbol{x},t)$ is of the form (c) and outside it is of the form (b) in (4.26). The second phase of self-dressing is concluded at time $t = t_1' + \frac{2a}{c}$ when the expanding causality sphere fills the whole volume within the shell of charges. Note that the form of the field $\boldsymbol{E}_c(\boldsymbol{x},t)$ at the surface of the shell of charges switches abruptly from (b) to (c) at time $t = \frac{2a}{c}$.

Summing up, the self-dressing of the shell of charges can be described in terms of two causality spheres, the first of which has radius $R_+ = c(t-t_1') + a$ and the second of which has radius $R_- = |c(t-t_1') - a| < R_+$. For $x > R_+$, $\boldsymbol{E}_c(\boldsymbol{x},t)$ vanishes. For $R_- < x < R_+$, $\boldsymbol{E}_c(\boldsymbol{x},t)$ is of the form (b) in (4.26). For $x < R_-$, $\boldsymbol{E}_c(\boldsymbol{x},t)$ vanishes if $t - t_1' < \frac{a}{c}$ and it is of the form (c) if $t - t_1' > \frac{a}{c}$.

4.3. Spherical volume

If the charge density is distributed uniformly within a sphere of radius a, $f(k) = 3j_1(ka)/ka$. Thus

$$\frac{1}{L^3}\sum_{\boldsymbol{k}} f(k)\cos[\boldsymbol{k}\cdot\boldsymbol{x} + \omega_k(t'-t)] = \frac{3}{2\pi^2 a}\frac{1}{x}\int_0^\infty \mathrm{d}k\, j_1(ka)\sin kx \cos ck(t'-t)$$
$$= \frac{1}{x}\frac{1}{2\pi^2}\sqrt{\frac{9\pi}{8a^3}}\int_0^\infty \mathrm{d}k\, k^{-1/2}J_{3/2}(ka)\{\sin k[x + c(t'-t)] + \sin k[x - c(t'-t)]\}. \tag{4.27}$$

The last integral can be done with the help of the tables [46] and we have

$$\frac{1}{L^3}\sum_{\boldsymbol{k}} f(k)\cos[\boldsymbol{k}\cdot\boldsymbol{x} + \omega_k(t'-t)] = \frac{1}{x}\frac{1}{2\pi^2}\sqrt{\frac{9\pi}{8a^3}}\frac{1}{\sqrt{2a}}\frac{\Gamma(1/2)}{\Gamma(2)}$$
$$\times\left\{C_1^{1/2}\left[\frac{x + c(t'-t)}{2a}\right]\theta\,[a - x - c(t'-t)]\mathrm{sgn}[x + c(t'-t)]\right.$$
$$\left.+ C_1^{1/2}\left[\frac{x - c(t'-t)}{2a}\right]\theta\,[a - x + c(t'-t)]\mathrm{sgn}[x - c(t'-t)]\right\} \tag{4.28}$$

where $\Gamma(z)$ is the gamma-function and the Gegenbauer polynomial is defined as $C_1^{1/2}(z) = 2z$. Consequently (4.4) gives

$$\boldsymbol{E}_c(\boldsymbol{x},t) = -\frac{3}{a^3}q\frac{1}{x}\int_{t_1'}^{t}\mathrm{d}t'\,\dot{\boldsymbol{Q}}(t')\{|c(t-t') - x|\theta[a + c(t-t') - x]$$
$$+ |c(t-t') + x|\theta[a - c(t-t') - x]\}. \tag{4.29}$$

The analysis of this expression for $x < a$ is not straightforward. For $x > a$, however, (4.29) can be cast in the form

$$\boldsymbol{E}_c(\boldsymbol{x},t) = -\frac{3}{a^3}q\frac{1}{x}\int_{t_1'}^{t-\frac{x-a}{c}}\mathrm{d}t'\,\dot{\boldsymbol{Q}}(t')|c(t-t') - x|\theta[c(t-t_1') - (x-a)] \tag{4.30}$$

which shows that $\boldsymbol{E}_c(\boldsymbol{x},t)$ is also causal for the spherical volume of charge in the region external to the source.

We conclude this section by discussing the relation between $\boldsymbol{E}_c(\boldsymbol{x},t)$ and $F_{\mathrm{RR}}(t)$ for a spherically symmetric source. Writing (4.4) in the form

$$\boldsymbol{E}_c(\boldsymbol{x},t) = -\frac{2\pi}{L^3}q\sum_{\boldsymbol{k}} f(k)\left\{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\int_{t_1'}^{t}\mathrm{d}t'\,\dot{\boldsymbol{Q}}(t')\mathrm{e}^{\mathrm{i}\omega_k(t'-t)} + \mathrm{cc}\right\} \tag{4.31}$$

we define the quantity

$$\begin{aligned}\boldsymbol{F}_c(t) &= \int_{L^3} \mathrm{d}^3\boldsymbol{x}\, \rho(x)\boldsymbol{E}_c(\boldsymbol{x},t) = q\int_{L^3} \mathrm{d}^3\,\boldsymbol{x}\, F(x)\boldsymbol{E}_c(\boldsymbol{x},t) \\ &= -\frac{4\pi}{L^3}q^2\sum_{\boldsymbol{k}} f^2(k)\int_{t_1'}^{t} \mathrm{d}t'\,\dot{\boldsymbol{Q}}(t')\cos\omega_k(t'-t) \\ &= -\frac{2}{\pi}q^2\int_{t_1'}^{t}\mathrm{d}t'\,\dot{\boldsymbol{Q}}(t')\int_0^\infty \mathrm{d}k\,k^2 f^2(k)\cos\omega_k(t'-t) \qquad (4.32)\end{aligned}$$

where we have used (2.18) and the first of (2.19). Comparison of (4.32) with (2.20) shows that $\boldsymbol{F}_c(t)$, which is the force exerted on the source by the field $\boldsymbol{E}_c(\boldsymbol{x},t)$ during self-dressing, is simply related to the radiation-reaction force as

$$\boldsymbol{F}_c(t) = \tfrac{3}{2}F_{\mathrm{RR}}(t). \qquad (4.33)$$

This is an interesting expression since it connects in detail the form of $F_{\mathrm{RR}}(t)$ with the propagation of $\boldsymbol{E}_c(\boldsymbol{x},t)$. For example, in the case of the spherical shell the jump in $F_{\mathrm{RR}}(t)$ for $t = t_1' + \frac{2a}{c}$ present in (3.4) is manifestly due to the impact of the internal causality sphere of radius R_- on the shell of charge.

5. Summary and conclusions

We have used a canonical approach to obtain the coupled equations of motion for a spherically symmetric charge and for the electromagnetic field in the Coulomb gauge of classical electrodynamics, using the constraints of one-dimensional slow motion and in the absence of external forces. We have assigned initial conditions to the field, such that the transverse part of the electric field vanishes at a predetermined time t_1' and we have obtained the equations of motion for the charge. These equations depend on t_1' in such a way that the radiation-reaction force $F_{\mathrm{RR}}(t)$ vanishes at $t = t_1'$ and they reduce to the familiar equations of motion for $t_1' \to -\infty$. Turning to the transverse part of the field, we have argued that, given the initial conditions assumed, it describes a process of classical self-dressing starting from a bare charge configuration. We have obtained an expression for $\boldsymbol{E}_\perp(\boldsymbol{x},t)$ at times $t > t_1'$, which we have partitioned into a longitudinal field $\boldsymbol{E}_L(\boldsymbol{x},t)$ and into another auxiliary field $\boldsymbol{E}_c(\boldsymbol{x},t)$. We have shown that the latter field develops causally in all cases explicitly considered (i.e. point charge, spherical shell and spherical volume) and that it contains the same information as $\boldsymbol{E}_\perp(\boldsymbol{x},t)$. Such a reconstruction of the electromagnetic field seems in agreement with the recent remarks by Carati and Galgani [47]. Thus we have concentrated on the details of the time development of $\boldsymbol{E}_c(\boldsymbol{x},t)$ and we have been able to relate the form of the radiation-reaction force $F_{\mathrm{RR}}(t)$ with these details. Finally, we have obtained a very simple relation between $F_{\mathrm{RR}}(t)$ and the force exerted by $\boldsymbol{E}_c(\boldsymbol{x},t)$ on the charge.

These results seem to open a wide field of investigation. Here we mention only the more obvious issues. First, it should be noted that the Bohr–Rosenfeld analysis of the measurement of the amplitude of a quantum field is based on a gedanken apparatus in which the pointer is constituted by a rigid mobile charge which at $t = t_1'$ is completely neutralized by an identical fixed charge [15]. Thus the transverse electric field at $t = t_1'$ vanishes and the present analysis should apply with relatively minor modifications. Secondly, self-dressing of a bare source in QED leads to a transition between two different vacua, which seems to introduce irreversibility in the dynamics, by the emission of low-frequency photons [9, 10]. It seems appropriate to investigate if an analogous irreversibility exists in classical self-dressing, particularly in view of the ongoing controversy about the meaning of irreversibility in electrodynamics [48, 49].

Thirdly, self-dressing in QED is fully causal and electromagnetic forces have been shown to be transmitted only causally, although field correlations are not causal [50]. It would be interesting to check if this is also the case for classical electrodynamics. Fourthly, the expressions for the radiation-reaction force during self-dressing obtained here seem to indicate that during an initial time, shorter than the time taken by the light to traverse the source, the dynamics of the latter is governed by a set of rapidly varying parameters such as effective mass and damping coefficient. It could be interesting to investigate the consequences of this initial variability on the motion of the source which takes place at later times, in view of the possibility of detecting experimentally some effects of self-dressing. We intend to address these questions in the future.

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